

# Enumeration of Circuits and Minimal Forbidden Sets

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# Enumeration of Circuits and Minimal Forbidden Sets

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## Abstract

In resource-constrained scheduling, it is sometimes important to know all inclusion-minimal subsets of jobs that must not be scheduled simultaneously. These so-called minimal forbidden sets are given implicitly by a linear inequality system, and can be interpreted more generally as the circuits of a particular independence system. We present several complexity results related to computation, enumeration, and counting of the circuits of an independence system. On this account, we also propose a backtracking algorithm that enumerates all minimal forbidden sets for resource constrained scheduling problems.

*Key words:* Independence system, Circuit, Enumeration, Minimal forbidden set

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## 1 Introduction

Given a finite ground set  $V$ , an independence system is defined as a family  $\mathcal{I}$  of subsets of  $V$  with two properties. First,  $\emptyset \in \mathcal{I}$ , and second, any subset of any member of  $\mathcal{I}$  also belongs to  $\mathcal{I}$ . The sets in  $\mathcal{I}$  are called independent sets, and the inclusion-maximal independent sets are the *bases*  $\mathcal{B}$  of  $\mathcal{I}$ . The sets not in  $\mathcal{I}$  are called dependent sets, and inclusion-minimal dependent sets are the *circuits*  $\mathcal{C}$  of  $\mathcal{I}$ ; see also [6]. Given a membership oracle for such an independence system  $\mathcal{I}$ , we are interested in the problem to enumerate all circuits of  $\mathcal{I}$ . It is obvious that the output size may be exponential in terms of the size of  $V$ . The complexity is thus measured in terms of the size of both, in- and output. Given that  $|V| = n$ , and given a  $(\text{poly}(n))$  membership oracle for some collection of subsets  $\mathcal{S} \subseteq 2^V$ , we use the following definitions; see [4].

**Definition 1** *The enumeration problem for  $\mathcal{S}$  is solvable in polynomial total time if there exists an algorithm and a polynomial  $p(\cdot, \cdot)$ , such that the algorithm correctly outputs all members of  $\mathcal{S}$  in  $p(n, |\mathcal{S}|)$  time.*

Given a sub-collection  $\mathcal{X} \subseteq \mathcal{S}$ , the *increments problem* is the problem: either decide that  $\mathcal{X} = \mathcal{S}$ , otherwise output a new element in  $\mathcal{S} \setminus \mathcal{X}$ .

**Definition 2** *The enumeration problem for  $\mathcal{S}$  is solvable in incremental polynomial time if there exists an algorithm and a polynomial  $p(\cdot, \cdot)$ , such that the algorithm correctly solves the increments problem in  $p(n, |\mathcal{X}|)$  time, for any  $\mathcal{X} \subseteq \mathcal{S}$ .*

If the enumeration problem for  $\mathcal{S}$  is solvable in incremental polynomial time, it is also solvable in polynomial total time: Starting with  $\mathcal{X} = \emptyset$ , the iterative solution of the increments problem yields the complete collection  $\mathcal{S}$ , in time polynomial in  $n$  and  $|\mathcal{S}|$ . The reverse, however, need not be true.

## 2 The general case

The following theorem is well known; it is proved using a reduction from the NP-complete decision problem SATISFIABILITY [3].

**Theorem 3 ([5])** *Unless  $P=NP$ , there does not exist a polynomial total time algorithm that enumerates the bases  $\mathcal{B}$  of any independence system  $\mathcal{I}$ .*

Likewise, using a simple duality argument, we can show the following.

**Theorem 4** *Unless  $P=NP$ , there does not exist a polynomial total time algorithm that enumerates the circuits  $\mathcal{C}$  of any independence system  $\mathcal{I}$ .*

The proof uses the fact that the circuits of  $\mathcal{I}$  are the bases of the following, dual independence system:  $\mathcal{I}^D = \{W \subseteq V \mid V \setminus W \notin \mathcal{I}\}$ . These two theorems say that, unless  $P=NP$ , an algorithm cannot exist which solves the problem for *any* independence system  $\mathcal{I}$ . For particular realizations of the membership oracle of  $\mathcal{I}$  however, efficient enumeration algorithms may well exist.

## 3 Scheduling and linear inequality systems

In resource-constrained scheduling, the input consists of a set of partially ordered jobs  $(V, \prec)$  (the partial order representing the precedence constraints) and resource constraints. The latter are given through a number of resource types  $k$  with availabilities  $b_k$ , and resource requirements  $a_{kj}$  of these resource types for all jobs  $j \in V$ . If a subset  $S$  of jobs consumes more of a resource type than available, the respective jobs in  $S$  may not be processed in parallel. The subsets of jobs that may be processed in parallel define an inde-

pendence system. The circuits of this independence system are either pairs of (precedence-)related jobs,  $\{i, j\}$  with  $i \prec j$ , or the so-called *minimal forbidden sets* (minimal anti-chains of  $(V, \prec)$  which may not be processed in parallel). For several algorithmic purposes, e.g. in stochastic scheduling [8], a complete list of the minimal forbidden sets is required. This amounts to the computation of the circuits of the corresponding independence system. The membership oracle of this system is a linear inequality system  $Ax \leq b$ , where  $A, b$  contains one row in for each resource type  $k$ , and one row for each edge in the comparability graph of the partial order  $(V, \prec)$ . The circuits are the minimally infeasible  $\{0, 1\}$ -vectors for  $Ax \leq b$ ; they are the incidence vectors of either pairs of (precedence-)related jobs, or minimal forbidden sets.

Inspired by [2], and using a reduction from the NP-complete decision problem INDEPENDENT SET in graphs [3], we show:

**Theorem 5** *Unless  $P=NP$ , there does not exist a polynomial total time algorithm that enumerates the minimally infeasible  $\{0, 1\}$ -vectors of an arbitrary linear inequality system  $Ax \leq b$ .*

In fact, in [2] it is shown that the decision version of the corresponding increments problem is NP-complete. What is interesting is the fact that the ‘dual’ problem is apparently much easier: It follows from [2] that the increments problem for the maximally feasible  $\{0, 1\}$ -vectors of an arbitrary linear inequality system  $Ax \leq b$  can be solved in quasi-polynomial time, hence there is also a quasi-polynomial total time algorithm for this problem. (Such a result is not likely for the problem considered here, because then all NP-hard problems could be solved in quasi-polynomial time.)

In terms of scheduling, Theorem 5 immediately yields:

**Corollary 6** *Unless  $P=NP$ , there does not exist a polynomial total time algorithm that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem.*

It turns out that even the computation of the *number* of minimal forbidden sets is a hard problem, because we can show:

**Theorem 7** *The problem to compute the number of minimally infeasible (maximally feasible)  $\{0, 1\}$ -vectors of an arbitrary linear inequality system  $Ax \leq b$  is #P-complete.*

The proof uses a reduction from the problem to compute a maximum cardinality anti-chain of a partial order. While this problem is polynomially solvable, the associated counting problem is known to be #P-complete [7].

Nevertheless, for practical purposes we implemented a simple backtracking al-

gorithm that lists the minimal forbidden sets for any instance of the resource-constrained project scheduling problem. In general, this algorithm can have an exponential running time in terms of in- and output of the problem. Yet, empirically it improves considerably upon a divide-and-conquer algorithm previously suggested in [5,1]; see [9]. Moreover, we can show that the algorithm is efficient for an important special case.

**Proposition 8** *There exists an incremental polynomial time (hence also a polynomial total time algorithm) that enumerates the minimal forbidden sets for any instance of the resource-constrained project scheduling problem, given that the number of resource types is 1.*

We refer to [9] for several other results related to enumeration and computation of minimal forbidden sets, as well as detailed computational results.

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